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COINCIDENCE POINT AND COMMON FIXED POINT FOR WEAKLY COMPATIBLE MAPPINGS IN METRIC SPACES

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ABSTRACT

Recently, Jaroslaw Gornicki(Fixed Point Theorem and Appl. 2017:9, 2017) introduced a new concept of $\mathcal{F}$-expansion mappings and proved a fixed point theorem for such mappings. Following this direction of research, in this paper, some coincidence point and common fixed point theorems has been proved for weakly compatible $\mathcal{F}$-expanding mappings.

Keywords- *Coincidence point; Common fixed point;* $\mathcal{F}$*-expanding mapping.* ***MSC 2000:*** *54H25****;*** *47H10.*

I. INTRODUCTION

In 2012, Wardowski [1] introduced the notion of $\mathcal{F}$-contractions and generalized the famous Banach contarction principle.He proved that an $\mathcal{F}$-contraction mapping on a complete metric space has a unique fixedpoint. An example of Wardowski [1] shows that sucha generalization of Banach contraction principle is a proper generalization. Afterwards, several authors extended and generalized this interesting result in various directions, see, e.g., [2], [3], [5], [6], [7], [9]. In this sequle,in 2017, Batra [8] proved an extension of the above mentioned result by presenting a common fixed point result for two commuting mappings on a complete metric space such that one of them is $\mathcal{F}$-dominated by the other. In the same year, Gornicki[4] introduced a new concept of $\mathcal{F}$-expansion mappings on a complete matric space with unique fixed point.

In this article, an extension of the results of Jaroslaw Gornicki[4] and Batra [8] has been worked out by presenting a coincidence point result and a common fixed point result for two weakly compatible mappings on a complete metric space such that one of them is $\mathcal{F}$-dominated by the other.

II. PRILIMINARIES

Throughout this paper $\mathbb{R}$ and $\mathbb{N}$ will denote the set of all real and set of all natural numbers respectively.

Definition 1 [8]: Let (X,d)be a complete metric space and $\mathcal{F}$ be the family of all functions $F:(0,\infty)\to\mathbb{R}$ such that:

(F1) F is strictly increasing, i.e., $F(\alpha)<F(\beta)$ for all $\alpha,\beta\in(0,\infty)$ and $\alpha<\beta$;
(F2) for each sequence $\{\alpha_n\}\subset(0,\infty)$,$\lim_{n\to\infty}\alpha_n=0$ if and only if $\lim_{n\to\infty}F(\alpha_n)=-\infty$
(F3) there exists a real number $k\in(0,1)$ such that $\lim_{\alpha\to0^+}\alpha^k F(\alpha)=0$.

Example 2[1]: Define $F:(0,\infty)\to\mathbb{R}$ by

(i) $F(\alpha)=\ln(\alpha)$
(ii) $F(\alpha)=\ln(\alpha)+\alpha$.

Then, $F\in\mathcal{F}$.

For more examples of such functionsreader is referred to [1].

Definition 3[4]: Let (X,d) be a metric space and $f:X\to X$ be a mapping. Then f is called expanding if it satisfies the following condition: there exists $\lambda>1$ such that

$$d(fx,fy)\geq\lambda d(x,y)\ \forall\ x,y\in X.$$

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Jarosalw Gornicki [4] generalized the exapanding mappings by defining the $\mathcal{F}$-expanding mappings as follows:

Definition 4[4]: Let (X,d) be a metric space. A mapping $f: X \to X$ is called $\mathcal{F}$-expandingif there exist $F \in \mathcal{F}$ and $t > 0$ such that for all $x, y \in X$,

$$d(x,y) > 0 \ \Rightarrow F(d(fx,fy)) \geq F(d(x,y)) + t. \qquad (1)$$

Remark 5: If one take $F(\alpha) = \ln(\alpha)$ for all $\alpha > 0$, then (1) reduces in the following form:

for all $x, y \in X$, $d(fx,fy) \geq \lambda d(x,y)$, where $\lambda = e^t > 1$.

Therefore, the exapanding mappings are a perticular case of $\mathcal{F}$- exapanding mappings.

Definition 6[5]: Let$X$ be a nonempty set,$f: X \to X$ and $g: X \to X$.If $w = fx = gx$ for some$x \in X$, then x is called a coicidence point of f and g and w is called the corresponding point of coincidence of $f$and g.

III. MAIN RESULT

This section contains the main results of this paper. First, we introduce some notions which will be needed in the sequel.

Definition 1: Let (X,d) be a metric space and the mappings $f: X \to X$ and $g: X \to X$ are satisfying the following property: there exists a number $\lambda > 1$ such that for all $x, y \in X$ we have

$$d(fx,fy) \geq \lambda d(gx,gy).$$

Then, the mapping f is called a g-expanding mapping.

Note that, an expanding mapping is a perticular case of g-expanding mapping (when $g = I_X$, the identity mapping of X), but a g- expanding mapping need not be an expanding mapping as shown in the following example.

Example 2: Let $X = \mathbb{R}$ and $d$be the usual metric on X, i.e

$$d(x,y) = |x - y| \ \forall \ x, y \in X.$$

Define two mappings $f, g: X \to X$ by:

$$f(x) = \begin{cases} \frac{x}{2} \text{if } x \in [0,1]; \\ 2x \text{ otherwise} \end{cases} \text{ and } g(x) = \begin{cases} \frac{x}{3} \text{if } x \in [0,1]; \\ x \ \text{ otherwise.} \end{cases}$$

Then, f is g-expanding mapping with $\lambda = \frac{3}{2}$. On the other hand, f is niether a contraction nor an expanding mapping.

Definition 3: Suppose, for $F \in \mathcal{F}$, the self mappings $f: X \to X$ and $g: X \to X$ satisfy the following property: there exist a number $t > 0$ such thatfor all $x, y \in X$

$$d(fx,fy) > 0, d(gx,gy) > 0 \ \Rightarrow \ F\big(d(fx,fy)\big) \geq F\big(d(gx,gy)\big) + t. \qquad (2)$$

Then, the mapping f is called an $\mathcal{F}$-g-expanding mapping.

When we consider in (2) the different types of the mapping $F \in \mathcal{F}$, then we obtain avariety of expanding mappings.Consider the following examples:

Example 4: Let $F : (0, \infty) \to \mathbb{R}$ be given by $F(\alpha) = \ln(\alpha)$, clearly F satisfies all the three condition (F1), (F2) and(F3) for any real number $k \in (0, 1)$ and for themappings $f: X \to X$ and $g: X \to X$ condition (2) is reduces into the following: there existst $t > 0$ such thatfor all $x, y \in X$

$$d(fx,fy) \geq e^t d(gx,gy).$$

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Example 5: Let $F : (0,\infty) \to \mathbb{R}$ be given by $F(\alpha) = \ln(\alpha) + \alpha$, clearly F satisfies all the three condition (F1), (F2) and(F3) for any real number $k \in (0,1)$ and for the mappings $f: X \to X$ and $g: X \to X$ condition (2) is reduces into the following:there exists$t > 0$ such thatfor all $x, y \in X$

$$d(fx, fy) \geq e^{t[d(gx,gy)-d(fx,fy)]} d(gx, gy).$$

Next theorem is a coincidence point result for an $\mathcal{F}$-g-expanding mapping f and the mapping g.

Theorem 6: Let (X, d) be a metric space and $f, g: X \to X$ be mapping suchthat $g(X) \subset f(X)$and $g(X)$is complete. Suppose, f is an $\mathcal{F}$-g-expanding mapping, then f and g have a point of coincidence.

Proof: Let $x_0 \in X$, then as $g(X) \subset f(X)$sothere exist $x_1 \in X$ such that$fx_1 = gx_0$. Similarly there exist $x_2 \in X$ such that $fx_2 = gx_1$. Proceeding in this manner, we get a sequence $\{y_n\} \in X$ such that $y_n = fx_{n+1} = gx_n \ \forall\, n \in \mathbb{N}$.
Let, $y_n = y_{n+1}$. Then we have:

$$fx_{n+1} = gx_n = fx_{n+2} = gx_{n+1}$$
$$\Rightarrow fx_{n+1} = gx_{n+1}.$$

This shows that x_{n+1} is a coincidence point of f and g.
Now, let us assume that$y_n \neq y_{n+1} \quad \forall\ n \in \mathbb{N}$. Then, as f is an $\mathcal{F}$-g-expanding mapping we have:

$$F\big(d(fx_n, fx_{n+1})\big) \geq F\big(d(gx_n, gx_{n+1})\big) + t$$
$$F\big(d(y_{n-1}, y_n)\big) \geq F\big(d(y_n, y_{n+1})\big) + t$$
$$\Rightarrow \quad F\big(d(y_n, y_{n+1})\big) \leq F\big(d(y_{n-1}, y_n)\big) - t$$
$$\Rightarrow \quad F\big(d(y_n, y_{n+1})\big) \leq F\big(d(y_{n-2}, y_{n-1})\big) - 2t.$$

Similarly, we get

$$F\big(d(y_n, y_{n+1})\big) \leq F\big(d(y_0, y_1)\big) - nt \qquad (3)$$
$$\Rightarrow \lim_{n\to\infty} F\big(d(y_n, y_{n+1})\big) \leq \lim_{n\to\infty} F\big(d(y_0, y_1)\big) - \lim_{n\to\infty} nt$$
$$\Rightarrow \lim_{n\to\infty} F\big(d(y_n, y_{n+1})\big) = -\infty$$
$$\Rightarrow \lim_{n\to\infty} d(y_n, y_{n+1}) = 0 \quad \text{by using (F2).}$$

Therefore, by (F3), there is a $k \in (0,1)$such that$\lim_{n\to\infty} d_n^k F(d_n) = 0$, where

$$d_n = d(y_n, y_{n+1}).$$

By equation (3) we have,

$$d_n^k F(d_n) \leq d_n^k F(d_0) - d_n^k . nt$$
$$\Rightarrow \lim_{n\to\infty} d_n^k [F(d_n) - F(d_0)] \leq -\lim_{n\to\infty} d_n^k . nt \leq 0$$
$$\Rightarrow \lim_{n\to\infty} d_n^k . n = 0.$$

From above equation, there exist $n_0 \in \mathbb{N}$ such that

$$|d_n^k . n - 0| < 1 \ \forall\ n > n_0$$
$$\Rightarrow d_n^k < \frac{1}{n} \ \forall\ n > n_0$$
$$\Rightarrow d_n < \frac{1}{n^{1/k}} \ \forall\ n > n_0.$$

Let us choose $m > n > n_0$, then

$$d(y_n, y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m) \leq d_n + d_{n+1} + \cdots + d_{m-1}$$
$$< \frac{1}{n^{1/k}} + \frac{1}{(n+1)^{1/k}} + \cdots + \frac{1}{(m-1)^{1/k}}$$
$$< \sum_{i=n}^{m-1} \frac{1}{i^{1/k}} \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}$$

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By the convergence of the series $\sum_{i=n}^{\infty}\frac{1}{i^{1/k}}$, and the above inequality we get $\{y_n\}=\{gx_n\}$is a Cauchy sequence in X. Since $g\,(X)$ is complete therefore there exist an element $y^*\in g\,(X)$such that

$$y_n \to y^* \in g\,(X) \subset f(X).$$

Let, $y_n \to y^* = fx^*$, where $x^*\in X$.
Since,

$$\begin{aligned} & F\big(d(fx^*,fx_n)\big) \ge F\big(d(gx^*,gx_n)\big)+t \\ \Rightarrow\ & F\big(d(gx^*,gx_n)\big) \le F\big(d(fx^*,fx_n)\big)-t \\ \Rightarrow\ & F\big(d(gx^*,y_n)\big) \le F\big(d(fx^*,y_{n-1})\big)-t \\ \Rightarrow\ & F\big(d(gx^*,y_n)\big) < F\big(d(fx^*,y_{n-1})\big) \ \ (as \ \ t>0) \\ \Rightarrow\ & d(gx^*,y_n) <\ d(y^*,y_{n-1}) \\ \Rightarrow\ & \lim_{n\to\infty} d(gx^*,y_n)=0 \\ \Rightarrow\ & y_n \to gx^* \text{as } n\to\infty. \end{aligned}$$

Therefore, by uniqueness of limit we have $gx^*=fx^*=y^*$. Thus, x^* is a coincidence point of f and g and y^* is the corrosponding point of coincidence of the mappings f and g.■

The following example shows that the above theorem ensures only the existence of point of coincidence of the mappings f and g, but not the existence of common fixed point of f and g.

Example 7: Let $X=\mathbb{R}$ and d is the usual metric on X, i.e., $d(x,y)=|x-y|$ for all $x,y\in X$. Define the mappings $f,g\colon X\to X$ by

$$fx = 2x, \qquad gx = 1\ \forall\ x\in X.$$

Then, it is easy to see that $g(X)\subset f(X)$, $g(X)=\{1\}$ is complete, and $d(gx,gy)=|1-1|=0$, therefore the condition (2) is satisfied trivially for all $F\in\mathcal{F}$. Note that, f and g have a coincidence point $\frac{1}{2}$ and the corresponding point of coincidence is 1. But, there is no common fixed point of f and g. ■

In the next theorem, a sufficient condition for the existence of common fixed point of f and g is provided.

Theorem 8: Suppose that all the conditions of Theorem 3.1 are satisfied. Then f and g have a point of coincidence. In addition, if f and g are weakly compatible then f and g have a unique common fixed point.

Proof: The existence of coincidence point x^* and the corresponding point of coincidencey^* follows from Theorem 1.Now, if f and g are weakly compatible, we have

$$gy^* = gfx^* = fgx^* = fy^* = w^* \text{ (say).}$$

Thus, $y^*$is a coincidence point and w^* is the corresponding point of coincidence of f and g.
If $fx^*=fy^*$, then $fy^*=y^*$ and $gy^*=y^*$, and so, y^* is a common fixed point of f and g. Similarly, if $gx^*=gy^*$ then again y^* is a common fixed point of f and g.

Now suppose that $fx^*\neq fy^*$ and $gx^*\neq gy^*$. Then by condition (2) we have:

$$\begin{aligned} & F\big(d(fy^*,y^*)\big) = F\big(d(fy^*,fx^*)\big) \\ \Rightarrow\ & F\big(d(w^*,y^*)\big) \ge F\big(d(gy^*,gx^*)\big)+t \\ & \qquad = F\big(d(w^*,y^*)\big)+t. \end{aligned}$$

Therefore, $F\big(d(w^*,y^*)\big)\ge F\big(d(w^*,y^*)\big)+t$. Since $t>0$ the above inequality yields a contradiction. Therefore, we must have $fy^*=y^*$ or $gy^*=y^*$, and so, as we have shown, in both the case y^* is a common fixed point of f and g.

For uniqueness of common fixed point, suppose $y^* \neq z^*$ are two common fixed point of f and g then by the definition of common fixed point $gy^* = fy^* = y^*$ and $gz^* = fz^* = z^*$.
Now,

$$F\big(d(y^*,z^*)\big) = F\big(d(fy^*,fz^*)\big) \geq F\big(d(gy^*,gz^*)\big) + t = \big(d(y^*,z^*)\big) + t.$$

Since $t > 0$ the above inequality yields a contradiction. Therefore, $y^* = z^*$,i.e.,the common fixed points of f and g is unique. ■

Remark 9: In this paper, we do not use the continuity of f or g, as well as, the commutative property of the mappings is not used for finding the common fixed point of mappings$f$ and g. While, e.g., Batra [8] uses both, the continuity, as well as, the commutativity of mappings f and g. On the other hand, Gornicki[4] assumed that the surjectivity of $\mathcal{F}$-expansion mappings and proved the existence of fixed point. In our results the mapping f need not to be surjective as shown in the following example.

Example 10: Let $X = \{1,2,3,4,5\}$ and d is the usual metric on X, i.e,

$$d(x,y) = |x - y| \ \forall \ x,y \in X.$$

Define two mappings $f, g: X \to X$ by:

$$f(x) = \begin{cases} x+1 & \text{if} \quad x \neq 5; \\ 5 & \text{if} \quad x = 5 \end{cases} \text{and} g(x) = 5 \ \forall \, x \in X.$$

Then, $g(X) = \{1\} \subset f(X)$, f is $\mathcal{F}$-g-expanding mapping for all $F \in \mathcal{F}$ and $g(X)$ is complete as $g(X)$ is singleton set. Also f and g are weakly compatible and $5 \in X$ is unique common fixed point point f and g. Note that the mapping f is not surjective, as, there is no $x \in X$ suchthat $fx = 1$. ■

Corollary 11: Let (X,d) be a complete metric space and $f: X \to X$ be surjective and $\mathcal{F}$-expanding. Then f has a unique fixed point.

Proof: Take $g = I_X$ in Theorem 3.8. Then, since f is surjective we have $g(X) = X \subset f(X)$ and, as, X is complete we have $g(X)$ is complete. Since f is $\mathcal{F}$-expanding and $g = I_X$, we have f is an $\mathcal{F}$-g-expanding mapping, also,f and g are weakly compatible. Thus, all the conditions of Theorem 3.8 are satisfied, and so, by Theorem 8f and g have a unique common fixed point. ■

Corollary 12: Let (X,d) be a complete metric space and $f: X \to X$ be surjective and expanding. Then f has a unique fixed point.

Proof: Take $F = \ln(\alpha)$ in the previous corollary, we obtain the required result. ■.

REFERENCES

[1] *D. Wardowski, Fixed points of a new type of contractive mappings in completemetric spaces, Fixed Point TheoryAppl. 2012:94 (2012).*

[2] *Hossein Piri, Poom Kumam, Some fixed point theorems concerning F-contraction in complete metric spaces, Springer, December 2014:210.*

[3] *I. Altun, G. Minak and H. Dağ,Multivalued F-contractions on complete metric spaces, Journal of Nonlinear and Convex Analysis, 2015.*

[4] *J. Gornicki, Fixed point theorems for F-expanding mappings, Fixed Point Theory and Application, Springer, 2017.*

[5] *M. Abbas, G. Jungck, Common fixed point results for non commuting mappings without continuity in cone matric spaces, ScienceDirect, 2008.*

[6] *M. Abbas, B. Ali and S. Romaguera, Fixed and periodic points of Generalized contractions in metric spaces, Springer, November 2013.*

[7] *N. Hussain and P. Salimi SalimiSuzuki-Wardowski Type Fixed Point Theorems for α-G-f-Contractions, Taiwanese Journal of Mathematics.*

[FRTSSDS- June 2018] ISSN 2348 – 8034
DOI: 10.5281/zenodo.1293825 Impact Factor- 5.070

[8] Rakesh Batra, Common fixed points for F-dominationg mappings, Global Journal of Pureand Applied Mathematics, 2017.

[9] S. Shukla and S. Radenović,Some Common Fixed Point Theorems forContraction TypeMappings in 0-Complete Partial Metric Spaces, Journal of MathematicsVolume 2013 (2013)..